Multiplication Operators on the Bloch Space of the Unit Disk

Graduate Seminar

Robert F. Allen
George Mason University
rallen2@gmu.edu

September 23, 2008

Acknowledgements

- This is joint work with my advisor Dr. Flavia Colonna.

Acknowledgements

- This is joint work with my advisor Dr. Flavia Colonna.

- R. Allen and F. Colonna, *Characterization of isometries and spectra of multiplication operators on the Bloch space*, preprint (http://arxiv.org/abs/0809.3278).

Motivation

Definition. (Isometry Problem) Given a Banach space (of analytic functions) X, what are the **isometries** on X? That is, what are the linear operators $T : X \to X$ such that

$$||Tx|| = ||x|| \, ,$$

for all $x \in X$?

Definition. (Isometry Problem) Given a Banach space (of analytic functions) X, what are the **isometries** on X? That is, what are the linear operators $T : X \to X$ such that

$$||Tx|| = ||x|| \, ,$$

for all $x \in X$?

The Isometry Problem was first investigated by Banach in 1932, when he proved that the surjective isometries on $C(Q)$, the space of continuous real-valued functions on a compact metric space Q, are of the form

$$Tf = \psi(f \circ \varphi),$$

where $|\psi| \equiv 1$ and φ is a homeomorphism of Q onto itself.

Definition. For $0 < p < \infty$, the **Hardy Space** H^p consists of analytic functions f on the open unit disk $\mathbb{D}$ such that

$$||f||_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \, d\theta \right)^{1/p} < \infty.$$

Isometries on H^p

Definition. For $0 < p < \infty$, the **Hardy Space** H^p consists of analytic functions f on the open unit disk $\mathbb{D}$ such that

$$||f||_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \, d\theta \right)^{1/p} < \infty.$$

Theorem. (Forelli, 1964) For $p \neq 2$, T is an isometry on H^p if and only if

$$Tf = \psi(f \circ \varphi),$$

for **some** ψ and φ.

Definition. For $0 < p < \infty$, the **Bergman Space** A^p consists of analytic functions f on $\mathbb{D}$ such that

$$||f||_p = \left(\int_{\mathbb{D}} |f(z)|^p \; dA \right)^{1/p} < \infty.$$

Definition. For $0 < p < \infty$, the **Bergman Space** A^p consists of analytic functions f on $\mathbb{D}$ such that

$$||f||_p = \left(\int_{\mathbb{D}} |f(z)|^p \; dA \right)^{1/p} < \infty.$$

Theorem. (Kolaski, 1982) Let $0 < p < \infty$, $p \neq 2$. Then $T : A^p \to A^p$ is a **surjective** linear isometry if and only if T has the form

$$Tf = \lambda(\varphi')^{2/p}(f \circ \varphi)$$

where φ is an automorphism of $\mathbb{D}$ and λ is a constant of modulus 1.

Weighted Composition Operators seem to be at the heart of the Isometry Problem, and thus are an important operator to study.

The Moral of the Story

Weighted Composition Operators seem to be at the heart of the Isometry Problem, and thus are an important operator to study.

Issues:

1. Weighted Composition Operators are difficult to study.

Weighted Composition Operators seem to be at the heart of the Isometry Problem, and thus are an important operator to study.

Issues:

1. Weighted Composition Operators are difficult to study.
2. In order to get an understanding of the behavior of the weighted composition operators, it is helpful to study the Multiplication and Composition Operators independently.

The Moral of the Story

Weighted Composition Operators seem to be at the heart of the Isometry Problem, and thus are an important operator to study.

Issues:

1. Weighted Composition Operators are difficult to study.
2. In order to get an understanding of the behavior of the weighted composition operators, it is helpful to study the Multiplication and Composition Operators independently.
3. The composition operators have been greatly studied, where as the multiplication operators have not.

The Moral of the Story

Weighted Composition Operators seem to be at the heart of the Isometry Problem, and thus are an important operator to study.

Issues:

1. Weighted Composition Operators are difficult to study.
2. In order to get an understanding of the behavior of the weighted composition operators, it is helpful to study the Multiplication and Composition Operators independently.
3. The composition operators have been greatly studied, where as the multiplication operators have not.

In order to understand the weighted composition operators better, we must first understand the multiplication operators.

The Bloch Space

Definition of the Bloch Space

Definition. An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be **Bloch** provided

$$\beta_f := \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

The **Bloch space**, defined as $\mathcal{B} = \{f \in H(\mathbb{D}) : \beta_f < \infty\}$, is a Banach space under the norm

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f.$$

Definition of the Bloch Space

Definition. An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be **Bloch** provided

$$\beta_f := \sup_{z \in \mathbb{D}} (1 - |z|^2)\,|f'(z)| < \infty.$$

The **Bloch space**, defined as $\mathcal{B} = \{f \in H(\mathbb{D}) : \beta_f < \infty\}$, is a Banach space under the norm

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f.$$

Theorem. (Schwarz-Pick Lemma) Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be analytic. Then for $z \in \mathbb{D}$,

$$(1 - |z|^2)\,|f'(z)| \leq 1 - |f(z)|^2.$$

If $f(z)$ is a conformal automorphism of $\mathbb{D}$, then equality holds; otherwise the inequality is strict for all $z \in \mathbb{D}$.

Consequences of the Schwarz-Pick Lemma

- All polynomials are Bloch.

Consequences of the Schwarz-Pick Lemma

- All polynomials are Bloch.
- All bounded analytic functions are Bloch, with

$$||f||_{\mathcal{B}} \leq 2\,||f||_{\infty}\,.$$

Consequences of the Schwarz-Pick Lemma

- All polynomials are Bloch.
- All bounded analytic functions are Bloch, with

$$||f||_{\mathcal{B}} \leq 2\,||f||_{\infty}\,.$$

- If $f \in \mathcal{B}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, then

$$\beta_{f\circ\varphi} \leq \beta_f\,.$$

Moreover, if φ is a conformal automorphism of $\mathbb{D}$, then equality holds.

Definition. Let $\mathcal{B}_*$ denote the set of Bloch functions that fix the origin, that is

$$\mathcal{B}_* = \{f \in \mathcal{B} : f(0) = 0\}.$$

The Isometry Problem

Definition. Let $\mathcal{B}_*$ denote the set of Bloch functions that fix the origin, that is

$$\mathcal{B}_* = \{f \in \mathcal{B} : f(0) = 0\}.$$

Theorem. (Cima & Wogen, 1980) Let $T : \mathcal{B}_* \to \mathcal{B}_*$ be a surjective isometry. Then there exists a conformal automorphism φ of $\mathbb{D}$ and a $\lambda \in \partial\mathbb{D}$ so that

$$T(f) = \lambda(f \circ \varphi) - \lambda f(\varphi(0)),$$

for all $f \in \mathcal{B}_*$.

Definition. A Bloch function f is called a **pointwise multiplier** of the Bloch space if

$$f\mathcal{B} \subseteq \mathcal{B}.$$

Pointwise Multipliers

Definition. A Bloch function f is called a **pointwise multiplier** of the Bloch space if

$$f\mathcal{B} \subseteq \mathcal{B}.$$

If $\psi : \mathbb{D} \to \mathbb{C}$ is analytic, we define the **multiplication operator** with symbol ψ on the Bloch space as

$$M_\psi(f) = \psi f,$$

for $f \in \mathcal{B}$.

Definition. A Bloch function f is called a **pointwise multiplier** of the Bloch space if

$$f\mathcal{B} \subseteq \mathcal{B}.$$

If $\psi : \mathbb{D} \longrightarrow \mathbb{C}$ is analytic, we define the **multiplication operator** with symbol ψ on the Bloch space as

$$M_\psi(f) = \psi f,$$

for $f \in \mathcal{B}$.

We see a very nice connection between functional analysis and operator theory in that ψ is a pointwise multiplier of the Bloch space if and only if $M_\psi : \mathcal{B} \longrightarrow \mathcal{B}$ is a bounded operator.

Previous Results

The boundedness of the multiplication operator was characterized independently by Arazy and Brown & Shields.

Boundedness

The boundedness of the multiplication operator was characterized independently by Arazy and Brown & Shields.

Theorem. [Arazy, 1982] An analytic function $\psi : \mathbb{D} \to \mathbb{C}$ induces a bounded multiplication operator M_ψ on the Bloch space if and only if ψ is bounded and

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |\psi'(z)| \log \frac{1 + |z|}{1 - |z|} < \infty.$$

The boundedness of the multiplication operator was characterized independently by Arazy and Brown & Shields.

Theorem. [Arazy, 1982] An analytic function $\psi : \mathbb{D} \to \mathbb{C}$ induces a bounded multiplication operator M_ψ on the Bloch space if and only if ψ is bounded and

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\,|\psi'(z)| \log \frac{1+|z|}{1-|z|} < \infty.$$

Theorem. [Brown & Shields, 1991] The analytic function $\psi : \mathbb{D} \to \mathbb{C}$ induces a bounded multiplication operator M_ψ on the Bloch space if and only if ψ is bounded and

$$|\psi'(z)| = O\left(\frac{1}{(1-|z|)\log\frac{1}{1-|z|}}\right).$$

Definition. An operator $T : X \to Y$ between Banach spaces is called **compact** if for every bounded sequence $\{x_n\} \subset X$, the image $\{Tx_n\}$ has compact closure in Y.

Compactness

Definition. An operator $T : X \to Y$ between Banach spaces is called **compact** if for every bounded sequence $\{x_n\} \subset X$, the image $\{Tx_n\}$ has compact closure in Y.

In their 2001 paper, Ohno and Zhao characterized the bounded and compact weighted composition operators on the Bloch space.

Definition. An operator $T : X \to Y$ between Banach spaces is called **compact** if for every bounded sequence $\{x_n\} \subset X$, the image $\{Tx_n\}$ has compact closure in Y.

In their 2001 paper, Ohno and Zhao characterized the bounded and compact weighted composition operators on the Bloch space.

As a special case, Ohno and Zhao deduced a characterization of the compact multiplication operators on the Bloch space.

Compactness

Definition. An operator $T : X \to Y$ between Banach spaces is called **compact** if for every bounded sequence $\{x_n\} \subset X$, the image $\{Tx_n\}$ has compact closure in Y.

In their 2001 paper, Ohno and Zhao characterized the bounded and compact weighted composition operators on the Bloch space.

As a special case, Ohno and Zhao deduced a characterization of the compact multiplication operators on the Bloch space.

Theorem. [Ohno & Zhao, 2001] The multiplication operator M_ψ is compact on $\mathcal{B}$ if and only if $\psi \equiv 0$.

My Research

Operator Norm Estimates

Definition. Let ψ be a Bloch function on $\mathbb{D}$. Define

$$\sigma_\psi = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\,|\psi'(z)|\log\frac{1+|z|}{1-|z|}.$$

Definition. Let ψ be a Bloch function on $\mathbb{D}$. Define

$$\sigma_\psi = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\,|\psi'(z)|\log\frac{1+|z|}{1-|z|}.$$

Theorem. [A. & Colonna, 2008] Suppose ψ is an analytic function on $\mathbb{D}$ inducing a bounded multiplication operator M_ψ on $\mathcal{B}$. Then

$$\max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\} \le ||M_\psi|| \le \max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_\psi\}.$$

Operator Norm Estimates

Definition. Let ψ be a Bloch function on $\mathbb{D}$. Define

$$\sigma_\psi = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\,|\psi'(z)|\log\frac{1+|z|}{1-|z|}.$$

Theorem. [A. & Colonna, 2008] Suppose ψ is an analytic function on $\mathbb{D}$ inducing a bounded multiplication operator M_ψ on $\mathcal{B}$. Then

$$\max\left\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\right\} \le ||M_\psi|| \le \max\left\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_\psi\right\}.$$

To achieve these estimates, we actually established estimates on the weighted composition operator first, and from them deduced the above estimates.

If ψ is a constant function of modulus 1, then M_ψ is an isometry on $\mathcal{B}$ since

$$||M_\psi(f)||_{\mathcal{B}} = |\psi(0)|\,|f(0)| + \beta_{\psi f} = |f(0)| + \beta_f = ||f||_{\mathcal{B}}\,.$$

Isometries

If ψ is a constant function of modulus 1, then M_ψ is an isometry on $\mathcal{B}$ since

$$||M_\psi(f)||_{\mathcal{B}} = |\psi(0)|\,|f(0)| + \beta_{\psi f} = |f(0)| + \beta_f = ||f||_{\mathcal{B}}\,.$$

Theorem. [A. & Colonna, 2008] The multiplication operator M_ψ is an isometry on $\mathcal{B}$ if and only if ψ is a constant function of modulus 1.

Definition. If T is a bounded linear operator on a complex Banach space E, then the **spectrum** of T is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

where I is the identity operator.

Spectrum

Definition. If T is a bounded linear operator on a complex Banach space E, then the **spectrum** of T is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

where I is the identity operator.

Theorem. [A. & Colonna, 2008] Let ψ be the symbol of a bounded multiplication operator M_ψ on $\mathcal{B}$. Then

$$\sigma(M_\psi) = \overline{\psi(\mathbb{D})}.$$

Definition. If T is a bounded linear operator on a complex Banach space E, then the **spectrum** of T is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

where I is the identity operator.

Theorem. [A. & Colonna, 2008] Let ψ be the symbol of a bounded multiplication operator M_ψ on $\mathcal{B}$. Then

$$\sigma(M_\psi) = \overline{\psi(\mathbb{D})}.$$

Corollary. [A. & Colonna, 2008] Let ψ be the symbol of an isometric multiplication operator M_ψ on $\mathcal{B}$. Then

$$\sigma(M_\psi) = \{\psi(0)\}.$$

Future Developments

Multiplication Operators

- Establish estimates for the essential norm.

Multiplication Operators

- Establish estimates for the essential norm.
- Characterize the essential spectrum.

Where To Go From Here

Multiplication Operators

- Establish estimates for the essential norm.
- Characterize the essential spectrum.

Weighted Composition Operators

- Characterize the isometries.

Multiplication Operators

- Establish estimates for the essential norm.
- Characterize the essential spectrum.

Weighted Composition Operators

- Characterize the isometries.
- Characterize the spectrum.

Where To Go From Here

Multiplication Operators

- Establish estimates for the essential norm.
- Characterize the essential spectrum.

Weighted Composition Operators

- Characterize the isometries.
- Characterize the spectrum.
- Establish estimates for the essential norm.

Where To Go From Here

Multiplication Operators

- Establish estimates for the essential norm.
- Characterize the essential spectrum.

Weighted Composition Operators

- Characterize the isometries.
- Characterize the spectrum.
- Establish estimates for the essential norm.
- Characterize the essential spectrum.

General References

- S. Banach, *Theorie des operations lineares*, Chelsea, Warsaw, 1932.
- F. Forelli, *The isometries on* H^p, Canadian J. Math **16**, 1964, 721 - 728.
- C. J. Kolaski, *Isometries on weighted Bergman spaces*, Canad. J. Math. **34**, 1982, 910 - 915.

Bloch Space References

- R. Allen and F. Colonna, *Characterization of isometries and spectra of multiplication operators on the Bloch space*, preprint (http://arxiv.org/abs/0809.3278).
- J. Arazy, *Multipliers of Bloch Functions*, University of Haifa Mathematics Publications **54**, 1982.
- L. Brown and A. L. Shields, *Multipliers and cyclic vectors in the Bloch space*, Michigan Math. J. **38**, 1991, 141 - 146.
- J. Cima and W. Wogen, *On isometries on the Bloch space*, Illinois J. Math **24**, 1980, 313 - 316.
- S. Ohno and Ruhan Zhao, *Weighted composition operators on the Bloch space*, Bull. Austral. Math. Soc **63**, 2001, 177-185.